

**Sixth Term Examination Papers**

**MATHEMATICS 2**

**WEDNESDAY 25 JUNE 2008**

**9470**

Morning

Time: 3 hours

Additional materials: Answer paper  
Graph paper  
Formulae booklet

**Candidates may not use electronic calculators**



**INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, Centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

**INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

You will be assessed on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with Mathematical Formulae and Tables.

**Electronic calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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**This question paper consists of 7 printed pages and 1 blank page.**

## Section A: Pure Mathematics

- 1 A sequence of points  $(x_1, y_1), (x_2, y_2), \dots$  in the cartesian plane is generated by first choosing  $(x_1, y_1)$  then applying the rule, for  $n = 1, 2, \dots$ ,

$$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + a, 2x_n y_n + b + 2),$$

where  $a$  and  $b$  are given real constants.

- (i) In the case  $a = 1$  and  $b = -1$ , find the values of  $(x_1, y_1)$  for which the sequence is constant.
- (ii) Given that  $(x_1, y_1) = (-1, 1)$ , find the values of  $a$  and  $b$  for which the sequence has period 2.
- 2 Let  $a_n$  be the coefficient of  $x^n$  in the series expansion, in ascending powers of  $x$ , of

$$\frac{1+x}{(1-x)^2(1+x^2)},$$

where  $|x| < 1$ . Show, using partial fractions, that either  $a_n = n + 1$  or  $a_n = n + 2$  according to the value of  $n$ .

Hence find a decimal approximation, to nine significant figures, for the fraction  $\frac{11\,000}{8181}$ .  
[You are not required to justify the accuracy of your approximation.]

- 3 (i) Find the coordinates of the turning points of the curve  $y = 27x^3 - 27x^2 + 4$ . Sketch the curve and deduce that  $x^2(1-x) \leq 4/27$  for all  $x \geq 0$ .  
Given that each of the numbers  $a, b$  and  $c$  lies between 0 and 1, prove by contradiction that at least one of the numbers  $bc(1-a), ca(1-b)$  and  $ab(1-c)$  is less than or equal to  $4/27$ .
- (ii) Given that each of the numbers  $p$  and  $q$  lies between 0 and 1, prove that at least one of the numbers  $p(1-q)$  and  $q(1-p)$  is less than or equal to  $1/4$ .

4 A curve is given by

$$x^2 + y^2 + 2axy = 1,$$

where  $a$  is a constant satisfying  $0 < a < 1$ . Show that the gradient of the curve at the point  $P$  with coordinates  $(x, y)$  is

$$-\frac{x + ay}{ax + y},$$

provided  $ax + y \neq 0$ . Show that  $\theta$ , the acute angle between  $OP$  and the normal to the curve at  $P$ , satisfies

$$\tan \theta = a|y^2 - x^2|.$$

Show further that, if  $\frac{d\theta}{dx} = 0$  at  $P$ , then:

(i)  $a(x^2 + y^2) + 2xy = 0$ ;

(ii)  $(1 + a)(x^2 + y^2 + 2xy) = 1$ ;

(iii)  $\tan \theta = \frac{a}{\sqrt{1 - a^2}}$ .

5 Evaluate the integrals

$$\int_0^{\pi/2} \frac{\sin 2x}{1 + \sin^2 x} dx \quad \text{and} \quad \int_0^{\pi/2} \frac{\sin x}{1 + \sin^2 x} dx.$$

Show, using the binomial expansion, that  $(1 + \sqrt{2})^5 < 99$ . Show also that  $\sqrt{2} > 1.4$ . Deduce that  $2^{\sqrt{2}} > 1 + \sqrt{2}$ . Use this result to determine which of the above integrals is greater.

6 A curve has the equation  $y = f(x)$ , where

$$f(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of  $f(x)$ .

(ii) Determine all values of  $x$  in the interval  $-\pi \leq x \leq \pi$  for which  $f(x) = 0$ . Find a value of  $x$  in this interval at which the curve touches the  $x$ -axis without crossing it.

(iii) Find the value or values of  $x$  in the interval  $0 \leq x \leq 2\pi$  for which  $f(x) = 2$ .

- 7 (i) By writing  $y = u(1 + x^2)^{\frac{1}{2}}$ , where  $u$  is a function of  $x$ , find the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = xy + \frac{x}{1 + x^2}$$

for which  $y = 1$  when  $x = 0$ .

- (ii) Find the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = x^2y + \frac{x^2}{1 + x^3}$$

for which  $y = 1$  when  $x = 0$ .

- (iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = x^{n-1}y + \frac{x^{n-1}}{1 + x^n}$$

for which  $y = 1$  when  $x = 0$ , where  $n$  is an integer greater than 1.

- 8 The points  $A$  and  $B$  have position vectors  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, relative to the origin  $O$ . The points  $A$ ,  $B$  and  $O$  are not collinear. The point  $P$  lies on  $AB$  between  $A$  and  $B$  such that

$$AP : PB = (1 - \lambda) : \lambda.$$

Write down the position vector of  $P$  in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\lambda$ . Given that  $OP$  bisects  $\angle AOB$ , determine  $\lambda$  in terms of  $a$  and  $b$ , where  $a = |\mathbf{a}|$  and  $b = |\mathbf{b}|$ .

The point  $Q$  also lies on  $AB$  between  $A$  and  $B$ , and is such that  $AP = BQ$ . Prove that

$$OQ^2 - OP^2 = (b - a)^2.$$

## Section B: Mechanics

- 9 In this question, use  $g = 10 \text{ m s}^{-2}$ .

In cricket, a fast bowler projects a ball at  $40 \text{ m s}^{-1}$  from a point  $h \text{ m}$  above the ground, which is horizontal, and at an angle  $\alpha$  above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of  $20 \text{ m}$  from the point of projection.

- (i) Determine, in terms of  $h$ , the two possible values of  $\tan \alpha$ .  
Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.
- (ii) State the range of values of  $h$  for which the bowler projects the ball below the horizontal.
- (iii) In the case  $h = 2.5$ , give an approximate value in degrees, correct to two significant figures, for  $\alpha$ . You need not justify the accuracy of your approximation.

[You may use the small-angle approximations  $\cos \theta \approx 1$  and  $\sin \theta \approx \theta$ .]

- 10 The lengths of the sides of a rectangular billiards table  $ABCD$  are given by  $AB = DC = a$  and  $AD = BC = 2b$ . There are small pockets at the midpoints  $M$  and  $N$  of the sides  $AD$  and  $BC$ , respectively. The sides of the table may be taken as smooth vertical walls.

A small ball is projected along the table from the corner  $A$ . It strikes the side  $BC$  at  $X$ , then the side  $DC$  at  $Y$  and then goes directly into the pocket at  $M$ . The angles  $BAX$ ,  $CXY$  and  $DYM$  are  $\alpha$ ,  $\beta$  and  $\gamma$  respectively. On each stage of its path, the ball moves with constant speed in a straight line, the speeds being  $u$ ,  $v$  and  $w$  respectively. The coefficient of restitution between the ball and the sides is  $e$ , where  $e > 0$ .

- (i) Show that  $\tan \alpha \tan \beta = e$  and find  $\gamma$  in terms of  $\alpha$ .
- (ii) Show that  $\tan \alpha = \frac{(1+2e)b}{(1+e)a}$  and deduce that the shot is possible whatever the value of  $e$ .
- (iii) Find an expression in terms of  $e$  for the fraction of the kinetic energy of the ball that is lost during the motion.

11 A wedge of mass  $km$  has the shape (in cross-section) of a right-angled triangle. It stands on a smooth horizontal surface with one face vertical. The inclined face makes an angle  $\theta$  with the horizontal surface. A particle  $P$ , of mass  $m$ , is placed on the inclined face and released from rest. The horizontal face of the wedge is smooth, but the inclined face is rough and the coefficient of friction between  $P$  and this face is  $\mu$ .

- (i) When  $P$  is released, it slides down the inclined plane at an acceleration  $a$  relative to the wedge. Show that the acceleration of the wedge is

$$\frac{a \cos \theta}{k + 1}.$$

To a stationary observer,  $P$  appears to descend along a straight line inclined at an angle  $45^\circ$  to the horizontal. Show that

$$\tan \theta = \frac{k}{k + 1}.$$

In the case  $k = 3$ , find an expression for  $a$  in terms of  $g$  and  $\mu$ .

- (ii) What happens when  $P$  is released if  $\tan \theta \leq \mu$ ?

## Section C: Probability and Statistics

- 12** In the High Court of Farnia, the outcome of each case is determined by three judges: the ass, the beaver and the centaur. Each judge decides its verdict independently. Being simple creatures, they make their decisions entirely at random. Past verdicts show that the ass gives a guilty verdict with probability  $p$ , the beaver gives a guilty verdict with probability  $p/3$  and the centaur gives a guilty verdict with probability  $p^2$ .

Let  $X$  be the number of guilty verdicts given by the three judges in a case. Given that  $E(X) = 4/3$ , find the value of  $p$ .

The probability that a defendant brought to trial is guilty is  $t$ . The King pronounces that the defendant is guilty if at least two of the judges give a guilty verdict; otherwise, he pronounces the defendant not guilty. Find the value of  $t$  such that the probability that the King pronounces correctly is  $1/2$ .

- 13** Bag  $P$  and bag  $Q$  each contain  $n$  counters, where  $n \geq 2$ . The counters are identical in shape and size, but coloured either black or white. First,  $k$  counters ( $0 \leq k \leq n$ ) are drawn at random from bag  $P$  and placed in bag  $Q$ . Then,  $k$  counters are drawn at random from bag  $Q$  and placed in bag  $P$ .

- (i) If initially  $n - 1$  counters in bag  $P$  are white and one is black, and all  $n$  counters in bag  $Q$  are white, find the probability in terms of  $n$  and  $k$  that the black counter ends up in bag  $P$ .

Find the value or values of  $k$  for which this probability is maximised.

- (ii) If initially  $n - 1$  counters in bag  $P$  are white and one is black, and  $n - 1$  counters in bag  $Q$  are white and one is black, find the probability in terms of  $n$  and  $k$  that the black counters end up in the same bag.

Find the value or values of  $k$  for which this probability is maximised.

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